

WEIGHT POSETS ASSOCIATED WITH GRADINGS OF SIMPLE LIE ALGEBRAS, WEYL GROUPS, AND ARRANGEMENTS OF HYPERPLANES

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1. INTRODUCTION

The set of weights of a finite-dimensional representation of a reductive Lie algebra has a natural poset structure (“weight poset”). Studying certain combinatorial problems related to antichains in weight posets, we realised that the best setting is provided by the representations associated with \mathbb{Z} -gradings of simple Lie algebras [13]. This article, which can be regarded as a sequel to [13], is devoted to a general theory of ideals (antichains) in the corresponding weight posets. Although the subject has interesting representation-theoretic aspects, we work here almost exclusively in the combinatorial setup. Specifically, our main object is going to be a \mathbb{Z} -graded root system.

Let V be an n -dimensional Euclidean space, with inner product (\cdot, \cdot) , and let Δ be an irreducible, crystallographic root system spanning V . We refer to [2, 7] for basic definitions and properties of root systems. Let Δ^+ be a set of positive roots and $\Pi = \{\alpha_1, \dots, \alpha_n\}$ the set of simple roots in Δ^+ . The usual partial order “ \preceq ” in Δ^+ is defined by the requirement that γ covers μ if and only if $\gamma - \mu \in \Pi$. A \mathbb{Z} -grading of Δ is a disjoint union $\Delta = \bigsqcup_{i \in \mathbb{Z}} \Delta(i)$ such that if $\gamma_1 \in \Delta(i_1)$, $\gamma_2 \in \Delta(i_2)$, and $\gamma_1 + \gamma_2$ is a root, then $\gamma_1 + \gamma_2 \in \Delta(i_1 + i_2)$. Then $\Delta(0)$ is a root system in its own sense. We always assume that Δ^+ is *compatible* with \mathbb{Z} -grading, which means that

$$(1.1) \quad \Delta^+ = \Delta(0)^+ \sqcup \Delta(1) \sqcup \Delta(2) \sqcup \dots,$$

where $\Delta(0)^+$ is a set of positive roots in $\Delta(0)$. Then $\Pi = \bigsqcup_{i \geq 0} \Pi(i)$, where $\Pi(i) = \Pi \cap \Delta(i)$, and $\Pi(0)$ is a set of simple roots for $\Delta(0)$. Each $\Delta(i)$, $i \geq 1$, can be regarded as a sub-poset of Δ^+ , and we are primarily interested in the poset $\Delta(1)$.

Let $\mathcal{J}_-(\Delta(1))$ be the set of lower (= order) ideals in $\Delta(1)$. We relate $\mathcal{J}_-(\Delta(1))$ to certain elements in the Weyl group W of Δ and certain hyperplane arrangements inside the Coxeter arrangement of Δ . The Weyl group of $\Delta(0)$, $W(0)$, is a parabolic subgroup of W . Let W^0 be the set of minimal length coset representatives for $W/W(0)$. It is known that

$$(1.2) \quad W^0 = \{w \in W \mid w(\alpha) \in \Delta^+ \quad \forall \alpha \in \Delta(0)^+\},$$

see [7, 1.10]. Let $N(w) = \{\gamma \in \Delta^+ \mid -w(\gamma) \in \Delta^+\}$ be the *inversion set* of $w \in W$ and $w \mapsto \ell(w) = \#N(w)$ the length function on W . By a classical result of Kostant [8, Prop. 5.10],

2010 *Mathematics Subject Classification.* 06A07, 17B20, 20F55.

Key words and phrases. Root system, graded Lie algebra, lower ideal, Coxeter arrangement.

$M \subset \Delta^+$ is the inversion set of some w if and only if both M and $\Delta^+ \setminus M$ are closed under addition. Such an M is said to be *bi-convex*.

Our basic results on the lower ideals in $\Delta(1)$ and related elements of W^0 are presented in Section 3. It is readily seen that if $w \in W^0$, then $I_w := N(w) \cap \Delta(1)$ is a lower ideal of $\Delta(1)$, which yields the map

$$\tau : W^0 \rightarrow \mathcal{J}_-(\Delta(1)), \quad w \mapsto \tau(w) := I_w.$$

For any $I \in \mathcal{J}_-(\Delta(1))$, we construct two extreme bi-convex subsets of Δ^+ that belong to $\bigcup_{k \geq 1} \Delta(k)$ and whose 1-component is I , see Theorems 3.3 and 3.4. This implies that τ is onto and $\tau^{-1}(I)$ contains a unique element of minimal and of maximal length. These two elements of W are said to be the *minimal* and the *maximal elements* of I , denoted $w_{I,\min}$ and $w_{I,\max}$, respectively. Furthermore, we observe that $\tau^{-1}(I)$ is an interval w.r.t. the weak Bruhat order “ \leq ” on W^0 ; that is, $\tau^{-1}(I) = \{w \in W^0 \mid w_{I,\min} \leq w \leq w_{I,\max}\}$, see Theorem 3.6.

Let W_{\min}^0 (resp. W_{\max}^0) be the subset of W^0 that consists of the minimal (resp. maximal) elements of all lower ideals. We provide a characterisation of each subset that does not refer to lower ideals. Set $\Delta(\geq k) = \sqcup_{j \geq k} \Delta(j)$ and $\Delta(\leq k) = \sqcup_{j \leq k} \Delta(j)$. Then

$$\begin{aligned} W_{\min}^0 &= \{w \in W^0 \mid w^{-1}(\alpha) \in \Delta(\geq -1) \text{ for all } \alpha \in \Pi\}, \text{ (Theorem 3.7);} \\ W_{\max}^0 &= \{w \in W^0 \mid w^{-1}(\alpha) \in \Delta(\leq 1) \text{ for all } \alpha \in \Pi\}, \text{ (Theorem 3.8).} \end{aligned}$$

We also point out a connection between an involution on $\mathcal{J}_-(\Delta(1))$, involution on W^0 , and the subset W_{\min}^0 and W_{\max}^0 (Proposition 3.9).

As an application of our minimal/maximal elements, we describe the antichains related to the lower ideals. Let $\min(M)$ and $\max(M)$ denote the minimal and maximal elements of a subset M w.r.t. the poset structure of $\Delta(1)$. For $I \in \mathcal{J}_-(\Delta(1))$, one may consider two antichains: $\max(I)$ and $\min(\Delta(1) \setminus I)$. Given $\gamma \in \Delta(1)$, our result is that

- $\gamma \in \max(I)$ if and only if $w_{I,\min}(\gamma) \in -\Pi$, see Theorem 4.1;
- $\gamma \in \min(\Delta(1) \setminus I)$ if and only if $w_{I,\max}(\gamma) \in \Pi$, see Theorem 4.2.

Associated with Δ^+ and $\Delta(0)^+$, there are two open dominant chambers, $\mathcal{C}^o = \{v \in V \mid (v, \alpha) > 0 \ \forall \alpha \in \Pi\}$ and $\mathcal{C}(0)^o = \{v \in V \mid (v, \alpha) > 0 \ \forall \alpha \in \Pi(0)\}$. The chambers $w(\mathcal{C}^o)$, $w \in W$, are said to be *small*. Let \mathcal{H}_γ denote the hyperplane in V orthogonal to $\gamma \in \Delta^+$. The hyperplanes \mathcal{H}_γ with $\gamma \in \Delta(1)$ dissect $\mathcal{C}(0)^o$ into certain regions, and we prove that there is a natural bijection between $\mathcal{J}_-(\Delta(1))$ and the set of these regions. Moreover, if $\mathcal{R}_I^o \subset \mathcal{C}(0)^o$ is the open region corresponding to I , then $w_{I,\min}^{-1}(\mathcal{C}^o)$ is the unique small chamber in \mathcal{R}_I^o closest to \mathcal{C}^o and $w_{I,\max}^{-1}(\mathcal{C}^o)$ is the unique small chamber in \mathcal{R}_I^o farthest from \mathcal{C}^o (Theorem 5.1). This result prompts considering the hyperplane arrangement $\mathcal{A}_\Delta(0, 1) = \{\mathcal{H}_\gamma \mid \gamma \in \Delta(0)^+ \cup \Delta(1)\}$ in V .

It is well known that the whole Coxeter arrangement $\mathcal{A}_\Delta = \{\mathcal{H}_\gamma \mid \gamma \in \Delta^+\}$ is free and its exponents are just the usual exponents of W [10, Ch. 6]. We conjecture that the arrangement $\mathcal{A}_\Delta(0, 1)$ is also free and its exponents are determined by certain partition associated with $\Delta(0)^+ \cup \Delta(1)$ (Conjecture 5.3). Actually, this is a special case of a more general conjecture that is discussed in the Introduction of [16]. Moreover, by [16, Theorem 11.1], that general conjecture and hence our Conjecture 5.3 are true if Δ is classical or of type \mathbf{G}_2 . For $\gamma \in \Delta$, let $[\gamma : \alpha_i]$ be the coefficient of α_i in the expression of γ via the simple roots. The height of γ is $\text{ht}(\gamma) = \sum_{i=1}^n [\gamma : \alpha_i]$. We deduce from Conjecture 5.3 that

$$\#\mathcal{J}_-(\Delta(1)) = \prod_{\gamma \in \Delta(1)} \frac{\text{ht}(\gamma) + 1}{\text{ht}(\gamma)}.$$

This equality has also been proved in [13], by *ad hoc* methods, for the abelian and extra-special gradings of Δ (see Section 2.3 for their definitions).

An inspiring observation is that, to a great extent, the theory of lower ideals in $\Delta(1)$ is parallel (similar) to the theory of upper (= *ad-nilpotent*) ideals in the poset (Δ^+, \preceq) . The latter will be referred to as the *affine theory*, because it requires the use of the affine Weyl group \widehat{W} and the affine root system $\widehat{\Delta}$. We discuss this parallelism in Section 6.

In Appendix A, we give a case-free proof of an observation in [16, Prop. 3.1] to the effect that certain sequence associated with an upper ideal of Δ^+ is, actually, a partition. This fact is also needed for Conjecture 5.3.

Acknowledgements. Part of this work was done while I was able to use rich facilities of the Max-Planck Institut für Mathematik (Bonn).

2. WEIGHT POSETS AND GRADINGS OF SIMPLE LIE ALGEBRAS

Let (\mathcal{P}, \preceq) be a finite poset. A *lower* (resp. *upper*) *ideal* I is a subset of \mathcal{P} such that if $\mu \in I$ and $\nu \preceq \mu$ (resp. $\nu \succeq \mu$), then $\nu \in I$. Let $\mathcal{J}_-(\mathcal{P})$ be the set of lower ideals, $\mathcal{J}_+(\mathcal{P})$ the set of upper ideals, and $\mathfrak{An}(\mathcal{P})$ the set of antichains in \mathcal{P} . For any $M \subset \mathcal{P}$, let $\min(M)$ (resp. $\max(M)$) denote the set of minimal (resp. maximal) elements of M with respect to ' \preceq '. The following three maps set up bijections between the respective pairs of sets:

$$\begin{aligned} I \in \mathcal{J}_-(\mathcal{P}) &\mapsto \max(I) \in \mathfrak{An}(\mathcal{P}), & I \in \mathcal{J}_+(\mathcal{P}) &\mapsto \min(I) \in \mathfrak{An}(\mathcal{P}), \\ I \in \mathcal{J}_-(\mathcal{P}) &\mapsto I^c := \mathcal{P} \setminus I \in \mathcal{J}_+(\mathcal{P}). \end{aligned}$$

Both $\mathcal{J}_-(\mathcal{P})$ and $\mathcal{J}_+(\mathcal{P})$ are graded posets under inclusion, with the rank function $I \mapsto \#I$. The rank-generating function of either of them is

$$\mathcal{M}_{\mathcal{P}}(t) = \sum_{I \in \mathcal{J}_-(\mathcal{P})} t^{\#I}.$$

It is also called the \mathcal{M} -polynomial of \mathcal{P} in [13]. Clearly, $\mathcal{M}_{\mathcal{P}}(1) = \#\mathcal{J}_-(\mathcal{P}) = \#\mathfrak{An}(\mathcal{P})$.

2.1. Gradings of simple Lie algebras and root systems. Although we are primarily interested in combinatorics of posets related to \mathbb{Z} -gradings of root systems, it is instructive and helpful to keep in mind that a \mathbb{Z} -grading of Δ is an offspring of a \mathbb{Z} -grading of the corresponding simple Lie algebra \mathfrak{g} . This provides a broader perspective and adds some geometric flavour and intuition to one's considerations. (We refer to [19, Ch. 3, § 3] for generalities on gradings of semisimple Lie algebras.)

Let $\mathfrak{g} = \mathfrak{u}^- \oplus \mathfrak{t} \oplus \mathfrak{u}$ be a fixed triangular decomposition, where \mathfrak{t} is a Cartan subalgebra of \mathfrak{g} . The associated root system $\Delta(\mathfrak{g}, \mathfrak{t})$ is Δ , and $V = \mathfrak{t}_{\mathbb{R}}^*$ is the \mathbb{R} -span of Δ in \mathfrak{t}^* . If \mathfrak{g}_{γ} is the root space for $\gamma \in \Delta$, then $\mathfrak{u} = \bigoplus_{\gamma \in \Delta^+} \mathfrak{g}_{\gamma}$. Write s_{γ} for the reflection in W with respect to $\gamma \in \Delta$. Let θ be the *highest root* in Δ^+ . Recall that $\text{ht}(\theta) = h - 1$, where h is the *Coxeter number* of Δ .

Let $\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}(i)$ be a \mathbb{Z} -grading. Since any derivation of \mathfrak{g} is inner, we have $\mathfrak{g}(i) = \{x \in \mathfrak{g} \mid [\tilde{h}, x] = ix\}$ for a unique (semisimple) element $\tilde{h} \in \mathfrak{g}(0)$. The element \tilde{h} is said to be *defining* for the grading in question. Here $\mathfrak{g}(0)$ is the centraliser of \tilde{h} , hence a reductive Lie algebra. Without loss of generality, one may assume that $\tilde{h} \in \mathfrak{t}$ and $\alpha(\tilde{h}) \geq 0$ for all $\alpha \in \Pi$. Then $\mathfrak{t} \subset \mathfrak{g}(0)$, $\mathfrak{g}(0) = (\mathfrak{g}(0) \cap \mathfrak{u}^-) \oplus \mathfrak{t} \oplus (\mathfrak{g}(0) \cap \mathfrak{u})$ is a triangular decomposition of $\mathfrak{g}(0)$, and

$$\mathfrak{u} = (\mathfrak{g}(0) \cap \mathfrak{u}) \oplus \mathfrak{g}(1) \oplus \mathfrak{g}(2) \oplus \dots$$

If $\Delta(i) = \{\gamma \in \Delta \mid \gamma(\tilde{h}) = i\}$, then $\Delta(i)$ is the set of roots of $\mathfrak{g}(i)$, and $\Delta = \bigsqcup_{i \in \mathbb{Z}} \Delta(i)$ is a compatible \mathbb{Z} -grading of Δ in the sense of Introduction, i.e., Eq. (1.1) holds. We also have $\Pi = \bigsqcup_{i \geq 0} \Pi(i)$, where $\Pi(i) = \{\alpha \in \Pi \mid \alpha(\tilde{h}) = i\}$, and $\Pi(0)$ is the set of simple roots in $\Delta(0)^+ = \Delta(0) \cap \Delta^+$.

Each $\mathfrak{g}(i)$ is a $\mathfrak{g}(0)$ -module, and therefore $\Delta(i)$ has a natural poset structure as the set of weights of a $\mathfrak{g}(0)$ -module. In case of compatible gradings, this *weight poset* structure on $\Delta(i)$ coincides with the restriction of ' \preceq ' to $\Delta(i)$, see [13, Remark 2.9]. More precisely, if $\gamma, \gamma' \in \Delta(i)$, then γ covers γ' if and only if $\gamma - \gamma' \in \Pi(0)$. Therefore, $\gamma' \preceq \gamma$ if and only if $\gamma - \gamma'$ is a nonnegative integer linear combination of $\Pi(0)$.

Set $\mathfrak{b}(0)^- = (\mathfrak{g}(0) \cap \mathfrak{u}^-) \oplus \mathfrak{t}$ and $\mathfrak{b}(0) = \mathfrak{t} \oplus (\mathfrak{g}(0) \cap \mathfrak{u})$. These are two opposite Borel subalgebras of $\mathfrak{g}(0)$. A link between combinatorics and geometry is provided by the following simple observation, which we do not pursue in this article.

Proposition 2.1. *There is a bijection between the lower (resp. upper) ideals of $(\Delta(1), \preceq)$ and the $\mathfrak{b}^-(0)$ -stable (resp. $\mathfrak{b}(0)$ -stable) subspaces of $\mathfrak{g}(1)$.*

Proof. If $I \in \mathcal{J}_-(\Delta(1))$ or $I \in \mathcal{J}_+(\Delta(1))$, then $\mathfrak{c}_I = \bigoplus_{\gamma \in I} \mathfrak{g}_{\gamma}$ is the corresponding $\mathfrak{b}^-(0)$ -stable or $\mathfrak{b}(0)$ -stable subspace of $\mathfrak{g}(1)$. The details are left to the reader. \square

2.2. Standard \mathbb{Z} -gradings. By [18, § 1.2, § 2.1], if one is interested in possible $\mathfrak{g}(0)$ -modules $\mathfrak{g}(i)$, and hence in possible posets $\Delta(i)$, then it suffices to consider the $\mathfrak{g}(0)$ -modules $\mathfrak{g}(1)$

for all semisimple \mathfrak{g} . (For $i > 1$, the problem is reduced to considering the induced \mathbb{Z} -grading of a certain semisimple subalgebra of \mathfrak{g} .) For this reason, it suffices to consider defining elements $\tilde{h} \in \mathfrak{t}$ such that $\alpha(\tilde{h}) \in \{0, 1\}$, i.e., $\Pi = \Pi(0) \sqcup \Pi(1)$. The corresponding \mathbb{Z} -gradings (of both \mathfrak{g} and Δ) are said to be *standard*. More precisely, if $\#\Pi(1) = k$, then we call it a *k-standard* grading. A standard \mathbb{Z} -grading can be represented by the Dynkin diagram of \mathfrak{g} , where the vertices in $\Pi(1)$ are coloured. If $\Pi(1) = \{\alpha_{i_1}, \dots, \alpha_{i_k}\}$, then the α_{i_j} 's are precisely the lowest weights of the simple $\mathfrak{g}(0)$ -modules in $\mathfrak{g}(1)$, the centre of $\mathfrak{g}(0)$ is k -dimensional, and $\mathfrak{g}(1)$ is a direct sum of k simple $\mathfrak{g}(0)$ -modules. In this case, the poset $\Delta(1)$ is the disjoint union of k subposets corresponding to the simple summands of $\mathfrak{g}(1)$. Therefore, all enumerative problems for $\Delta(1)$ reduce to 1-standard gradings.

The weight posets $\Delta(i)$, $i > 0$, can be visualised as follows. Let $\mathcal{H}(\Delta^+)$ be the *Hasse diagram* of (Δ^+, \preceq) . If $\gamma' - \gamma = \alpha \in \Pi$, then the edge connecting γ and γ' in $\mathcal{H}(\Delta^+)$ is said to be of *type* α . Given a standard \mathbb{Z} -grading of \mathfrak{g} , let us remove from $\mathcal{H}(\Delta^+)$ all the edges of types from $\Pi(1)$. This yields a disconnected graph. Each connected component of it is the Hasse diagram of either the set of positive roots of a simple factor of $\mathfrak{g}(0)$ (if it contains roots from $\Pi(0)$) or the weight poset of a simple $\mathfrak{g}(0)$ -module in some $\mathfrak{g}(i)$, $i > 0$. The set of weights of a simple $\mathfrak{g}(0)$ -module in some $\mathfrak{g}(i)$, $i \geq 1$, is precisely the set of roots γ with fixed values $[\gamma : \alpha]$ for all $\alpha \in \Pi(1)$, see e.g. [19, 3.5].

2.3. Special classes of \mathbb{Z} -gradings. In [13, Sect. 3, 4], we considered in details the following two classes of \mathbb{Z} -gradings of \mathfrak{g} and hence of Δ :

The abelian case: $\mathfrak{g} = \mathfrak{g}(-1) \oplus \mathfrak{g}(0) \oplus \mathfrak{g}(1)$.

Here $\mathfrak{g}(0) \oplus \mathfrak{g}(1)$ is a parabolic subalgebra and $\mathfrak{g}(1)$ is its abelian nilradical. In this case $\mathfrak{g}(1)$ is a simple $\mathfrak{g}(0)$ -module and therefore such a grading is 1-standard. If $\Pi(1) = \{\tilde{\alpha}\}$, then upon the identification of $\mathfrak{t}_{\mathbb{R}}$ and $\mathfrak{t}_{\mathbb{R}}^*$, the defining element \tilde{h} appears to be the minuscule fundamental weight $\varphi_{\tilde{\alpha}}^{\vee}$ of the dual root system Δ^{\vee} . As is well known, the admissible simple roots $\tilde{\alpha}$ are characterised by the property that $[\theta : \tilde{\alpha}] = 1$ [3, Ch. VIII, § 7, n°3].

The extra-special case: $\mathfrak{g} = \mathfrak{g}(-2) \oplus \mathfrak{g}(-1) \oplus \mathfrak{g}(0) \oplus \mathfrak{g}(1) \oplus \mathfrak{g}(2)$ and $\dim \mathfrak{g}(2) = 1$.

Any simple Lie algebra has a unique, up to conjugation, \mathbb{Z} -grading of this form, and without loss of generality, we may assume that $\Delta(2) = \{\theta\}$. Upon the identification of $\mathfrak{t}_{\mathbb{R}}$ and $\mathfrak{t}_{\mathbb{R}}^*$, the defining element \tilde{h} is recognised as the coroot θ^{\vee} . That is, $\Delta(i) = \{\gamma \in \Delta \mid (\gamma, \theta^{\vee}) = i\}$ and $W(0)$ is the stabiliser of θ (or θ^{\vee}) in W . Since here $\Pi(1) = \{\alpha \in \Pi \mid (\gamma, \theta^{\vee}) \neq 0\}$, we see that $\mathfrak{g}(1)$ is a simple $\mathfrak{g}(0)$ -module if and only if θ is a multiple of a fundamental weight, i.e., \mathfrak{g} is not of type A_n .

The following simple lemma is one of our main tools for inductive arguments in subsequent sections.

Lemma 2.2. *Suppose that the roots μ, ν_1, ν_2 have the property that $\nu_1 + \nu_2 \in \Delta$ and $\mu + \nu_1 + \nu_2 \in \Delta$. Then $\mu + \nu_1$ or $\mu + \nu_2$ is also a root.*

Proof. 1) If $(\mu + \nu_1 + \nu_2, \nu_1 + \nu_2) > 0$, then $(\mu + \nu_1 + \nu_2, \nu_1) > 0$ or $(\mu + \nu_1 + \nu_2, \nu_2) > 0$. Hence $\mu + \nu_2$ or $\mu + \nu_1$ is a root.

2) If $(\mu + \nu_1 + \nu_2, \nu_1 + \nu_2) \leq 0$, then $(\mu, \nu_1 + \nu_2) < 0$. Hence $(\mu, \nu_1) < 0$ or $(\mu, \nu_2) < 0$, i.e., again $\mu + \nu_1$ or $\mu + \nu_2$ is a root. \square

3. ELEMENTS OF W^0 ASSOCIATED WITH THE LOWER IDEALS IN $\Delta(1)$

In this section, $\Delta = \bigsqcup_{i \in \mathbb{Z}} \Delta(i)$ is a \mathbb{Z} -grading and $\Delta^+ = \Delta(0)^+ \cup \Delta(\geq 1)$. Recall that $I \in \mathcal{J}_-(\Delta(1))$ if and only if whenever $\gamma \in I$, $\mu \in \Delta(0)^+$, and $\gamma - \mu \in \Delta$, then $\gamma - \mu \in I$. Then $I^c := \Delta(1) \setminus I \in \mathcal{J}_+(\Delta(1))$. That is, if $\gamma \in I^c$, $\mu \in \Delta(0)^+$, and $\gamma + \mu \in \Delta$, then $\gamma + \mu \in I^c$.

For any $I \subset \Delta^+$, we set $I^1 = I$ and if $I^{k-1} \neq \emptyset$, then $I^k = (I + I^{k-1}) \cap \Delta$ for $k \geq 2$. Then $\langle I \rangle := \bigcup_{k \geq 1} I^k \subset \Delta^+$.

Lemma 3.1. $\langle I \rangle$ is a closed subset of Δ^+ .

Proof. Suppose that $\gamma_i \in I^{k_i}$, $i = 1, 2$ and $\gamma_1 + \gamma_2$ is a root. Our goal is to prove that $\gamma_1 + \gamma_2 \in I^{k_1+k_2}$. Without loss of generality, we may assume that $k_1 \leq k_2$. Arguing by induction, we assume that the required property holds for all (k'_1, k'_2) such that either $k'_1 + k'_2 < k_1 + k_2$ or $k'_1 + k'_2 = k_1 + k_2$ and $k'_1 < k_1$.

- If $k_1 = 1$, then $\gamma_1 + \gamma_2 \in I^{k_1+k_2}$ by the very definition of I^k .
- If $k_1 > 1$, then $\gamma_1 = \gamma'_1 + \gamma''_1$ with $\gamma'_1 \in I$ and $\gamma''_1 \in I^{k_1-1}$. By Lemma 2.2, we then have $\gamma'_1 + \gamma_2 \in \Delta$ or $\gamma''_1 + \gamma_2 \in \Delta$. Hence, by the induction assumption, $\gamma'_1 + \gamma_2 \in I^{k_2+1}$ or $\gamma''_1 + \gamma_2 \in I^{k_1+k_2-1}$; and in either case we also conclude that $\gamma'_1 + \gamma''_1 + \gamma_2 \in I^{k_1+k_2}$. \square

Now, let us turn to the case in which $I \subset \Delta(1)$. Then $I^k \subset \Delta(k)$ for all $k \geq 1$. Consequently, $\langle I \rangle \subset \Delta(\geq 1)$.

Proposition 3.2. If $I \in \mathcal{J}_-(\Delta(1))$, then $I^k \in \mathcal{J}_-(\Delta(k))$ for any $k \geq 1$. Likewise, if $I \in \mathcal{J}_+(\Delta(1))$, then $I^k \in \mathcal{J}_+(\Delta(k))$ for any $k \geq 1$.

Proof. Argue by induction on k and use Lemma 2.2. \square

Theorem 3.3. If $I \in \mathcal{J}_-(\Delta(1))$, then $\langle I \rangle$ is a bi-convex subset of Δ^+ .

Proof. By Lemma 3.1, $\langle I \rangle$ is closed. Set $(I^k)^c = \Delta(k) \setminus I^k$ for $k \geq 1$. Then

$$\overline{\langle I \rangle} := \Delta^+ \setminus \langle I \rangle = \Delta(0)^+ \cup I^c \cup (I^2)^c \cup \dots,$$

and our goal is to prove that $\overline{\langle I \rangle}$ is closed, too. Assuming that this is not the case, one can find $\mu', \mu'' \in \overline{\langle I \rangle}$ such that $\mu' + \mu'' \in \langle I \rangle$. Since $\Delta(0)^+$ is closed and each $(I^k)^c$ is an upper ideal (use Proposition 3.2!), one has to only consider the case in which neither μ' nor μ'' belong to $\Delta(0)^+$. Specifically, assume that $\mu' \in (I^i)^c$ and $\mu'' \in (I^j)^c$ with $i, j \geq 1$, but $\mu' + \mu'' \in I^{i+j}$. Arguing by induction, we may assume that $i + j$ is the smallest integer with such property. By the recursive definition of $\langle I \rangle$, one has

$$\mu' + \mu'' = \gamma_1 + \gamma_{i+j-1} \in I^{i+j},$$

where $\gamma_k \in I^k$. Since $(\mu' + \mu'', \gamma_1 + \gamma_{i+j-1}) > 0$, we may assume that, say, $(\mu', \gamma_1) > 0$ and hence $\nu := \mu' - \gamma_1 = \gamma_{i+j-1} - \mu'' \in \Delta(i-1)$. Now, there are two possibilities for i .

(a) $i = 1$. Then $\nu \in \Delta(0)$. If $\nu \in \Delta(0)^+$, then $\mu' = \gamma_1 + \nu \in I$. If $\nu \in \Delta(0)^-$, then $\mu'' = \gamma_j - \nu \in I^j$. In either case, this contradicts the assumption on μ', μ'' .

(b) $i > 1$. Then $\nu = \mu' - \gamma_1 \in \Delta(i-1) \subset \Delta^+$. If $\nu \in I^{i-1}$, then $\mu' \in I^i$, a contradiction. If $\nu \in (I^{i-1})^c$, then

$$(\mu' - \gamma_1) + \mu'' = \gamma_{i+j-1} \in I^{i+j-1},$$

which contradicts the minimality of $i + j$.

Thus, $\overline{\langle I \rangle}$ is closed, and we are done. \square

Theorem 3.4. *If $I \in \mathcal{J}_-(\Delta(1))$, then $\Delta(\geq 1) \setminus \langle I^c \rangle$ is a bi-convex subset of Δ^+ .*

Proof. All the necessary ideas are already contained in the previous proof.

1. The complement in Δ^+ of the indicated subset is $\langle I^c \rangle \cup \Delta(0)^+$. Here $\langle I^c \rangle = \bigcup_{k \geq 1} (I^c)^k$ is closed by Lemma 3.1, and, by Proposition 3.2, each $(I^c)^k$ is an upper ideal of $\Delta(k)$. Therefore, $\langle I^c \rangle \cup \Delta(0)^+$ is closed, too.

2. To prove that $\Delta(\geq 1) \setminus \langle I^c \rangle = \bigcup_{k \geq 1} (\Delta(k) \setminus (I^c)^k)$ is closed, one uses the fact that each $\Delta(k) \setminus (I^c)^k$ is a lower ideal and repeats *mutatis mutandis* the inductive argument of the previous proof. \square

By Theorem 3.3, there is a unique $w \in W$ such that $N(w) = \langle I \rangle$. In particular,

$$(\clubsuit) \quad N(w) \cap \Delta(1) = I.$$

Since $N(w) \subset \Delta(\geq 1)$, we also have $w \in W^0$. Furthermore, if $w' \in W^0$ also satisfies (\clubsuit) , then $N(w') \supset \langle I \rangle = N(w)$. Thus, w is the unique element of **minimal** length in W^0 such that the 1-component of $N(w)$ is I . We shall say that w is the *minimal element of I* and denote it by $w_{I,\min}$. Likewise, by Theorem 3.4, there is a unique $\tilde{w} \in W^0$ such that $N(\tilde{w}) = \Delta(\geq 1) \setminus \langle I^c \rangle$. Clearly, the 1-component of $N(\tilde{w})$ is I . Furthermore, if $w' \in W^0$ also satisfies (\clubsuit) , then $\Delta^+ \setminus N(w') \supset \langle I^c \rangle \cup \Delta(0)^+ = \Delta^+ \setminus N(\tilde{w})$. Thus, \tilde{w} is the unique element of **maximal** length in W^0 such that the 1-component of $N(\tilde{w})$ is I . For this reason, we say that \tilde{w} is the *maximal element of I* and denote it by $w_{I,\max}$.

Remark 3.5. It is readily seen that if $w \in W^0$, then $I_w := N(w) \cap \Delta(1)$ is a lower ideal in $\Delta(1)$. This provides the natural map $\tau : W^0 \rightarrow \mathcal{J}_-(\Delta(1))$, $w \mapsto I_w$. An offspring of Theorems 3.3 and 3.4 is that τ is onto and we have two sections $s_{\min}, s_{\max} : \mathcal{J}_-(\Delta(1)) \rightarrow W^0$ for τ , where $s_{\min}(I) = w_{I,\min}$ and $s_{\max}(I) = w_{I,\max}$.

Recall that the *weak Bruhat order* “ \leq ” on (any subset of) W is defined by the condition that $w \leq w'$ if and only if $N(w) \subset N(w')$. As a consequence of preceding results, we obtain the following interesting fact.

Theorem 3.6. *For any $I \in \mathcal{J}_-(\Delta(1))$, $\tau^{-1}(I)$ is an interval with respect to the weak Bruhat order in W^0 . Namely, $\tau^{-1}(I) = \{w \in W^0 \mid w_{I,\min} \leq w \leq w_{I,\max}\}$.*

Proof. If $w \in \tau^{-1}(I)$, then $N(w) \cap \Delta(1) = I$ and hence

$$N(w_{I,\min}) = \langle I \rangle \subset N(w) \subset \Delta(\geq 1) \setminus \langle I^c \rangle = N(w_{I,\max}),$$

in view of the definitions of $w_{I,\min}$ and $w_{I,\max}$. That is, $w_{I,\min} \leq w \leq w_{I,\max}$.

The other implication is obvious. \square

Definition 1. The set of *minimal elements* of W^0 is $W_{\min}^0 = \{w_{I,\max} \mid I \in \mathcal{J}_-(\Delta(1))\}$;

The set of *maximal elements* of W^0 is $W_{\max}^0 = \{w_{I,\max} \mid I \in \mathcal{J}_-(\Delta(1))\}$.

Our next aim is to provide alternative descriptions of the sets $W_{\min}^0 = s_{\min}(\mathcal{J}_-(\Delta(1)))$ and $W_{\max}^0 = s_{\max}(\mathcal{J}_-(\Delta(1)))$.

Theorem 3.7. $W_{\min}^0 = \{w \in W^0 \mid w^{-1}(\alpha) \in \Delta(\geq -1) \text{ for all } \alpha \in \Pi\}$.

Proof. (i) Suppose that $w = w_{I,\min}$ and $w^{-1}(\alpha) \in \Delta(-k)$ for some $\alpha \in \Pi$ and $k \geq 1$. More precisely, if $w^{-1}(\alpha) = -\gamma$, then $w(\gamma) = -\alpha$. Hence $\gamma \in I^k$. Assume that $k \geq 2$. Then $\gamma = \gamma' + \gamma''$ with $\gamma' \in I$ and $\gamma'' \in I^{k-1}$. Here we would obtain that $-\alpha = w(\gamma') + w(\gamma'')$ is a sum of two negative roots, which is absurd. Thus, $k \leq 1$.

(ii) Conversely, suppose that $w \in W^0$ has the property that $w^{-1}(\alpha) \in \Delta(\geq -1)$ for all $\alpha \in \Pi$. Set $I = N(w) \cap \Delta(1)$. Then $I \in \mathcal{J}_-(\Delta(1))$, because $w \in W^0$. Therefore $\langle I \rangle = N(w_{I,\min})$ and $N(w_I) \subset N(w)$. The last inclusion implies that $w = uw_{I,\min}$ for some $u \in W$ such that $\ell(w) = \ell(u) + \ell(w_{I,\min})$, see e.g. [16, Lemma 5.1]. Assume that $u \neq 1_W$. Then $w = s_\alpha u' w_{I,\min}$ for some $\alpha \in \Pi$ such that $\ell(u) = 1 + \ell(u')$ and therefore

$$N(w) = N(u' w_{I,\min}) \cup (u' w_{I,\min})^{-1}(\alpha).$$

Since $\ell(u' w_{I,\min}) = \ell(u') + \ell(w_{I,\min})$, we have $N(u' w_{I,\min}) \supset N(w_{I,\min}) \supset I$. Therefore $(u' w_{I,\min})^{-1}(\alpha) \in \Delta(k)$ and here $k \geq 2$. Then $w^{-1}(\alpha) = -(u' w_{I,\min})^{-1}(\alpha) \in \Delta(-k)$, which contradicts the assumption on w . Thus, $w = w_{I,\min}$, and we are done. \square

Theorem 3.8. $W_{\max}^0 = \{w \in W^0 \mid w^{-1}(\alpha) \in \Delta(\leq 1) \text{ for all } \alpha \in \Pi\}$.

Proof. The proof is similar to the previous one and left to the reader. \square

Below we point out a relationship between an involution on $\mathcal{J}_-(\Delta(1))$, involution on W^0 , and the subsets W_{\min}^0 and W_{\max}^0 . Let $w_0 \in W$ and $\tilde{w}_0 \in W(0)$ be the respective longest elements. It is easily seen that if $w \in W^0$, then $w_0 w \tilde{w}_0 \in W^0$. Therefore, the mapping

$$w \in W^0 \mapsto i(w) := w_0 w \tilde{w}_0 \in W^0$$

is a well-defined involution on W^0 , see [6]. For any $I \in \mathcal{J}_-(\Delta(1))$, we have defined the *dual lower ideal* I^* by $I^* = \tilde{w}_0(\Delta(1) \setminus I)$, see [13, Sect. 2]. Note that $\#I + \#I^* = \#\Delta(1)$.

Proposition 3.9. For any $w \in W^0$, we have

(i) $(I_w)^* = I_{i(w)}$.

(ii) $w \in W_{\min}^0$ if and only if $i(w) \in W_{\max}^0$. More precisely, $i(w_{I,\min}) = w_{I^*,\max}$.

Proof. (i) We have $N(w_0 w \tilde{w}_0) = \Delta^+ \setminus N(w \tilde{w}_0)$. Since $\ell(w \tilde{w}_0) = \ell(w) + \ell(\tilde{w}_0)$, one also has $N(w \tilde{w}_0) = N(\tilde{w}_0) \cup (\tilde{w}_0)^{-1} N(w) = \Delta(0)^+ \cup \tilde{w}_0(N(w))$ [16, Lemma 5.1]. Therefore, $N(w_0 w \tilde{w}_0) \cap \Delta(1) = \Delta(1) \setminus \tilde{w}_0(N(w) \cap \Delta(1))$. That is, $I_{i(w)} = \Delta(1) \setminus \tilde{w}_0(I_w) = (I_w)^*$.

(ii) Combine part (i), characterisations of W_{\min}^0 and W_{\max}^0 in Theorems 3.7, 3.8, and the following properties of the longest elements: w_0 takes Π to $-\Pi$; whereas \tilde{w}_0 takes each $\Delta(i)$ to itself and also $\Delta(0)^+$ to $-\Delta(0)^+$. \square

For any subset $S \subset W$, define its Poincaré polynomial by $S(t) = \sum_{w \in S} t^{\ell(w)} = \sum_{w \in S} t^{\#N(w)}$.

The celebrated Kostant-Macdonald identity [9] says that

$$(3.1) \quad W(t) = \prod_{\gamma \in \Delta^+} \frac{1 - t^{\text{ht}(\gamma)+1}}{1 - t^{\text{ht}(\gamma)}}.$$

In particular, $\#W = \prod_{\gamma \in \Delta^+} \frac{\text{ht}(\gamma)+1}{\text{ht}(\gamma)}$.

Example 3.10. Let $\Delta = \bigsqcup_{i=-1}^1 \Delta(i)$ be an abelian grading. Then Theorems 3.7 and 3.8 immediately imply that $W_{\min}^0 = W_{\max}^0 = W^0$. Therefore $\#\mathcal{J}_-(\Delta(1)) = \#W^0$. Furthermore, $i(w) = w$ if and only if $(I_w)^* = I_w$. For any parabolic subgroup $W(0) \subset W$, we have

$$\#\{w \in W^0 \mid i(w) = w\} = W^0(-1),$$

see [6, 14]. Therefore, in the abelian case, $W^0(-1)$ equals the number of self-dual lower ideals in $\Delta(1)$. This has already been proved in [17]. In the abelian case, $W^0(t)$ coincides with the rank-generating function for the poset of lower ideals, see e.g. [13, Sect. 3], i.e., $W^0(t) = \mathcal{M}_{\Delta(1)}(t)$ and thereby $\mathcal{M}_{\Delta(1)}(-1)$ is the number of self-dual lower ideals.

Remark 3.11. For the non-abelian \mathbb{Z} -gradings (i.e., if $\Delta(2) \neq \emptyset$), W_{\min}^0 and W_{\max}^0 are different proper subsets W^0 . Moreover, the polynomials $W_{\min}^0(t)$, $W_{\max}^0(t)$, and $\mathcal{M}_{\Delta(1)}(t)$, which have the same value at $t = 1$, are different. For the reader convenience, we compare explicit formulae for all these polynomials:

$$\begin{aligned} \mathcal{M}_{\Delta(1)}(t) &= \sum_{w \in W_{\min}^0} t^{\#(N(w) \cap \Delta(1))} = \sum_{w \in W_{\max}^0} t^{\#(N(w) \cap \Delta(1))}, \\ W_{\min}^0(t) &= \sum_{w \in W_{\min}^0} t^{\#N(w)}, \quad W_{\max}^0 = \sum_{w \in W_{\max}^0} t^{\#N(w)}. \end{aligned}$$

We have conjectured in [13, Conjecture 5.2] (and verified in many cases) that $\mathcal{M}_{\Delta(1)}(-1)$ yields the number of self-dual lower ideals in $\Delta(1)$ for **any** \mathbb{Z} -grading. That is, in a sense, $\mathcal{M}_{\Delta(1)}(t)$ is the most appropriate t -analogue of $\#\mathcal{J}_-(\Delta(1))$.

Example 3.12. Let $\Delta = \bigsqcup_{i=-2}^2 \Delta(i)$ be an extra-special grading. As $\Delta(2) = \{\theta\}$, it follows from Theorem 3.7 that, for $w \in W^0$, we have $w \in W_{\min}^0$ if and only if $w^{-1}(\alpha) \neq -\theta$ for all $\alpha \in \Pi$, i.e., $-w(\theta) \notin \Pi$. Likewise, by Theorem 3.8, $w \in W_{\max}^0$ if and only if $w(\theta) \notin \Pi$. Hence here $W^0 = W_{\min}^0 \cup W_{\max}^0$. As $W(0)$ is the stabiliser of θ in W , we have $W^0 \cdot \theta = W \cdot \theta$,

$\#W^0$ is the number of long roots in Δ , and the non-minimal (or non-maximal) elements of W^0 are parameterised by the set, Π_l , of long simple roots. Since the number of long roots is $\#\Pi_l \cdot h$ [2, Chap. VI, § 1.11, Prop. 33], this yields the equality $\#W_{\min}^0 = \#\Pi_l \cdot (h - 1)$. The last formula for the number of the ideals/antichains in $\Delta(1)$ was obtained earlier in [13, Theorem 4.2].

Remark. If a \mathbb{Z} -grading is neither abelian nor extra-special, then $W^0 \neq W_{\min}^0 \cup W_{\max}^0$.

Suppose now that the \mathbb{Z} -grading in question is 1-standard. More precisely, $\Pi = \Pi(0) \cup \Pi(1)$ and $\Pi(1) = \{\tilde{\alpha}\}$. For any $w \in W^0$, we look at the coefficient of $\tilde{\alpha}$ for the roots $w^{-1}(\alpha)$, $\alpha \in \Pi$. Namely, write

$$w^{-1}(\alpha) = k_\alpha(w)\tilde{\alpha} + \sum_{\alpha_i \in \Pi(0)} l_i(w)\alpha_i$$

and consider the mapping $\eta : W^0 \rightarrow \mathbb{Z}^n$, $\eta(w) = (k_\alpha(w))_{\alpha \in \Pi}$.

Theorem 3.13. (i) *The mapping η is injective;*

(ii) $\eta(W_{\min}^0) = \{(k_\alpha(w))_{\alpha \in \Pi} \mid k_\alpha(w) \geq -1 \text{ for all } \alpha \in \Pi\}$;

(iii) $\eta(W_{\max}^0) = \{(k_\alpha(w))_{\alpha \in \Pi} \mid k_\alpha(w) \leq 1 \text{ for all } \alpha \in \Pi\}$;

Proof. (i) Let $\{\varpi_\alpha^\vee\}_{\alpha \in \Pi}$ be the fundamental weights of the dual Lie algebra \mathfrak{g}^\vee corresponding to Π . In other words, $(\alpha, \varpi_\beta^\vee) = \delta_{\alpha\beta}$ for all $\alpha, \beta \in \Pi$, i.e., $\{\varpi_\alpha^\vee\}_{\alpha \in \Pi}$ is the dual basis to Π . Then $W(0)$ is the stabiliser of $\varpi_{\tilde{\alpha}}^\vee$ in W and all weights $w(\varpi_{\tilde{\alpha}}^\vee)$, $w \in W^0$, are different. We have $(w(\varpi_{\tilde{\alpha}}^\vee), \alpha) = (\varpi_{\tilde{\alpha}}^\vee, w^{-1}(\alpha)) = k_\alpha(w)$. Whence $w(\varpi_{\tilde{\alpha}}^\vee) = \sum_{\alpha \in \Pi} k_\alpha(w)\varpi_\alpha^\vee$.

(ii), (iii). This readily follows from Theorems 3.7 and 3.8, because $w^{-1}(\alpha) \in \Delta(i)$ if and only if $k_\alpha(w) = i$. \square

Remark. The above proof suggests to regard η as a mapping from W^0 to the lattice $\mathcal{L} = \{\sum_{\alpha \in \Pi} k_\alpha \varpi_\alpha^\vee \mid k_\alpha \in \mathbb{Z}\} \simeq \mathbb{Z}^n$ in V . Set also $\mathcal{C}_{\geq -1} = \{\sum_{\alpha \in \Pi} k_\alpha \varpi_\alpha^\vee \mid k_\alpha \geq -1 \forall \alpha \in \Pi\}$ and $\mathcal{C}_{\leq 1} = \{\sum_{\alpha \in \Pi} k_\alpha \varpi_\alpha^\vee \mid k_\alpha \leq 1 \forall \alpha \in \Pi\}$. Then Theorem 3.13 asserts that

$$\eta(W_{\min}^0) = W \cdot \varpi_{\tilde{\alpha}}^\vee \cap \mathcal{C}_{\geq -1} \text{ and } \eta(W_{\max}^0) = W \cdot \varpi_{\tilde{\alpha}}^\vee \cap \mathcal{C}_{\leq 1}.$$

Thus, the minimal or maximal elements of W^0 are in a natural one-to-one correspondence with certain subsets of the W -orbit of $\varpi_{\tilde{\alpha}}^\vee$.

Example 3.14. The abelian gradings are 1-standard and then $\varpi_{\tilde{\alpha}}^\vee$ is a minuscule fundamental weight of \mathfrak{g}^\vee . Then $(\varpi_{\tilde{\alpha}}^\vee, \gamma) \in \{-1, 0, 1\}$ for all $\gamma \in \Delta$ [3, Ch. VIII, § 7, n°3]. Consequently, the whole orbit $W \cdot \varpi_{\tilde{\alpha}}^\vee$ belongs to $\mathcal{C}_{\geq -1} \cap \mathcal{C}_{\leq 1}$. Here we again obtain that all elements of W^0 are both maximal and minimal, and therefore $\#\mathfrak{An}(\Delta(1)) = \#W^0$.

4. EXTREME ROOTS ASSOCIATED WITH THE LOWER IDEALS IN $\Delta(1)$

Recall that any lower (resp. upper) ideal of a poset \mathcal{P} is determined by its maximal (resp. minimal) elements. Below, we describe these extreme elements (roots) for the ideals in $\mathcal{P} = \Delta(1)$, using the corresponding minimal and maximal elements of W^0 .

Theorem 4.1. *If $I \in \mathcal{J}_-(\Delta(1))$ and $\gamma \in \Delta(1)$, then $\gamma \in \max(I)$ if and only if $w_{I,\min}(\gamma) \in -\Pi$.*

Proof. Write w for $w_{I,\min}$ in this proof. Recall that $\gamma \in I$ if and only if $w(\gamma) \in -\Delta^+$.

(i) If $\gamma \in I$ and $\gamma \notin \max(I)$, then $\gamma = \gamma' - \delta$ for some $\gamma' \in I$ and $\delta \in \Delta(0)^+$. Then $w(\gamma) = w(\gamma') - w(\delta)$ is a sum of negative roots.

(ii) Conversely, if $\gamma \in I$ and $w(\gamma) \notin -\Pi$, then $w(\gamma) = -\delta_1 - \delta_2$, where $\delta_i \in \Delta^+$. Hence $-w^{-1}(\delta_1) - w^{-1}(\delta_2) = \gamma \in \Delta(1)$. Set $\mu_i = -w^{-1}(\delta_i)$, so that $\gamma = \mu_1 + \mu_2$. Without loss of generality, we may assume that μ_2 is positive. Let us consider possible levels of μ_2 and consequences of that for γ .

(1) The case in which $\mu_2 \in \Delta(0)^+$ is impossible, since $w(\mu_2) = -\delta_2$ and $w \in W^0$.

(2) Suppose that $\mu_2 \in \Delta(1)$. Since $w(\mu_2)$ is negative, we have $\mu_2 \in I$. Furthermore, here $\mu_1 \in \Delta(0)$. As in (1), the case $\mu_1 \in \Delta(0)^+$ is impossible. Hence $-\mu_1 \in \Delta(0)^+$ and then $\gamma = \mu_1 + \mu_2 \prec \mu_2$, i.e., $\gamma \notin \max(I)$.

(3) Suppose that $\mu_2 \in \Delta(k)$, $k \geq 2$. Let us show that there is another decomposition $\gamma = \tilde{\mu}_1 + \tilde{\mu}_2$ such that $\tilde{\mu}_2 \in \Delta(\tilde{k})$ with $0 < \tilde{k} < k$.

Since $w(\mu_2)$ is negative, we have $\mu_2 \in I^k$ by the very definition of $w = w_{I,\min}$. Hence, $\mu_2 = \mu' + \mu''$, where $\mu' \in I^{k'}$, $\mu'' \in I^{k''}$, and $k' + k'' = k$. As $\gamma = \mu_1 + \mu' + \mu''$, we have $\mu_1 + \mu' \in \Delta$ or $\mu_1 + \mu'' \in \Delta$, see Lemma 2.2. By symmetry, it suffices to consider the first possibility. Then we set $\tilde{\mu}_1 = \mu_1 + \mu'$, $\tilde{\mu}_2 = \mu''$, and $\tilde{k} = k''$.

Thus, one can gradually descend to the case $\tilde{k} = 1$ and conclude using (2) that $\gamma \notin \max(I)$. \square

Theorem 4.2. *For $I \in \mathcal{J}_-(\Delta(1))$ and $\gamma \in \Delta(1)$, we have $\gamma \in \min(I^c)$ if and only if $w_{I,\max}(\gamma) \in \Pi$.*

Proof. This proof is similar (and “dual”) to the proof of Theorem 4.1. Write w for $w_{I,\max}$ in this proof. Recall that $\gamma \in I^c$ if and only if $w(\gamma) \in \Delta^+$.

(i) If $\gamma \in I^c \setminus \min(I^c)$, then $\gamma = \gamma' + \delta$ for some $\gamma' \in I^c$ and $\delta \in \Delta(0)^+$. Then $w(\gamma) = w(\gamma') + w(\delta)$ is a sum of positive roots.

(ii) Conversely, if $\gamma \in I^c$ and $w(\gamma) \notin \Pi$, then $w(\gamma) = \delta_1 + \delta_2$, where $\delta_i \in \Delta^+$. Hence $w^{-1}(\delta_1) + w^{-1}(\delta_2) = \gamma \in \Delta(1)$. Set $\mu_i = w^{-1}(\delta_i)$. Without loss of generality, we may assume that μ_2 is positive. Let us consider possible levels of μ_2 and consequences of that for γ .

(1) Suppose that $\mu_2 \in \Delta(0)^+$. Then $\mu_1 \in \Delta(1)$ and $w(\mu_1) \in \Delta^+$. Hence $\mu_1 \in I^c$ and $\gamma = \mu_1 + \mu_2 \notin \min(I^c)$.

(2) Suppose that $\mu_2 \in \Delta(1)$. Then $\mu_2 \in I^c$ and $\mu_1 \in \Delta(0)$.

– If μ_1 is positive, then again $\gamma = \mu_1 + \mu_2 \notin \min(I^c)$.

– The case in which $\mu_1 \in -\Delta(0)^+$ is impossible, since $w(\mu_1) = \delta_1$ and $w \in W^0$.

(3) Suppose that $\mu_2 \in \Delta(k)$, $k \geq 2$. Let us show that there is another decomposition $\gamma = \tilde{\mu}_1 + \tilde{\mu}_2$ such that $\tilde{\mu}_2 \in \Delta(\tilde{k})$ with $0 < \tilde{k} < k$.

Since $w(\mu_2) \in \Delta^+$, we have $\mu_2 \in (I^c)^k$ by the very definition of $w = w_{I, \max}$. Hence, $\mu_2 = \mu' + \mu''$, where $\mu' \in (I^c)^{k'}$, $\mu'' \in (I^c)^{k''}$, and $k' + k'' = k$. As $\gamma = \mu_1 + \mu' + \mu''$, we have $\mu_1 + \mu' \in \Delta$ or $\mu_1 + \mu'' \in \Delta$, see Lemma 2.2. By symmetry, it suffices to handle the first possibility. Then we set $\tilde{\mu}_1 = \mu_1 + \mu'$, $\tilde{\mu}_2 = \mu''$, and $\tilde{k} = k''$.

Thus, one can gradually descend to the case $\tilde{k} = 1$ and conclude using (2) that $\gamma \notin \max(I^c)$. \square

5. DOMINANT CHAMBERS AND ARRANGEMENTS OF HYPERPLANES

For $\gamma \in \Delta$, let \mathcal{H}_γ be the hyperplane in V orthogonal to γ . Then $\mathcal{A} = \{\mathcal{H}_\gamma \mid \gamma \in \Delta^+\}$ is the Coxeter arrangement associated with Δ . The connected components of $V \setminus (\bigcup_{\gamma \in \Delta^+} \mathcal{H}_\gamma)$ are called (open) *chambers*. Each chamber is an open simplicial cone in V , and W acts simply transitively on the set of chambers. The *dominant* open chamber is $\mathcal{C}^o = \{v \in V \mid (v, \alpha) > 0 \ \forall \alpha \in \Pi\}$. The closure of \mathcal{C}^o is denoted by \mathcal{C} . If $\mathcal{K}', \mathcal{K}''$ are two chambers, then the *distance* between them, $d(\mathcal{K}', \mathcal{K}'')$, is the number of hyperplanes in \mathcal{A} that separate them. As is well known, $d(\mathcal{C}, w(\mathcal{C})) = \ell(w)$. More precisely, the hyperplane \mathcal{H}_γ separates \mathcal{C} and $w(\mathcal{C})$ if and only if $\gamma \in N(w^{-1})$, see [2, Chap. VI, § 1, Prop. 17].

In this section, we will consider certain sub-arrangements of \mathcal{A}_Δ and their relationship to ideals/antichains in the poset $\Delta(1)$. The first of them is $\mathcal{A}_{\Delta(0)} = \{\mathcal{H}_\gamma \mid \gamma \in \Delta(0)^+\}$, the Coxeter arrangement associated with $\Delta(0)$. The corresponding *big* dominant chamber is $\mathcal{C}(0)^o = \{v \in V \mid (v, \alpha) > 0 \ \forall \alpha \in \Pi(0)\}$ and its closure is denoted by $\mathcal{C}(0)$. It follows readily from the definition of W^0 (see Eq. (1.2)), that $w \in W^0$ if and only if $w^{-1}(\mathcal{C}) \subset \mathcal{C}(0)$. In particular, the big dominant chamber $\mathcal{C}(0)$ is the union of $\#W^0$ “small” chambers.

Theorem 5.1.

(i) The hyperplanes \mathcal{H}_γ , $\gamma \in \Delta(1)$, dissect the cone $\mathcal{C}(0)$ into certain regions (cones) that are in a natural one-to-one correspondence with the ideals of $\Delta(1)$ (and we write \mathcal{R}_I^o for the open region corresponding to $I \in \mathcal{J}_-(\Delta(1))$);

(ii) if $w \in W_{\min}^0$, then $w^{-1}(\mathcal{C}^o)$ is the unique small chamber in $\mathcal{R}_{I_w}^o$ that is closest to \mathcal{C}^o ;

(iii) if $w \in W_{\max}^0$, then $w^{-1}(\mathcal{C}^o)$ is the unique small chamber in $\mathcal{R}_{I_w}^o$ that is farthest from \mathcal{C}^o ;

Proof. (i) Given $I \in \mathcal{J}_-(\Delta(1))$, define the open region (cone), \mathcal{R}_I^o , corresponding to I as follows:

$$\mathcal{R}_I^o = \{x \in \mathcal{C}(0)^o \mid (x, \gamma) > 0 \text{ if } \gamma \notin I \text{ \& } (x, \gamma) < 0 \text{ if } \gamma \in I\}.$$

Using the fact that $\tau : W^0 \rightarrow \mathcal{J}_-(\Delta(1))$ is onto, one immediately obtains that $\mathcal{R}_I^o \neq \emptyset$ for any I . Indeed, if $\tau(w) = I$, then \mathcal{H}_γ ($\gamma \in \Delta(1)$) separates \mathcal{C}^o and $w^{-1}(\mathcal{C}^o)$ if and only if $\gamma \in N(w) \cap \Delta(1) = I$. Therefore, $w^{-1}(\mathcal{C}^o) \subset \mathcal{R}_I^o$. Furthermore, any chamber $w^{-1}(\mathcal{C}^o)$, $w \in W^0$, belongs to some region \mathcal{R}_I^o , which means that the closed regions \mathcal{R}_I ($I \in \mathcal{J}_-(\Delta(1))$) exhaust the big dominant chamber $\mathcal{C}(0)$.

(ii),(iii) This follows from (i) and the fact that $w_{I,\min}$ (resp. $w_{I,\max}$) is the unique element of minimal (resp. maximal) length in $\tau^{-1}(I)$. \square

These properties suggest to consider the sub-arrangement $\mathcal{A}_\Delta(0, 1)$ of \mathcal{A}_Δ that contains only the hyperplanes \mathcal{H}_γ corresponding to $\gamma \in \Delta(0)^+ \cup \Delta(1)$. Set $\eta_i = \#\{\gamma \in \Delta(0)^+ \cup \Delta(1) \mid \text{ht}(\gamma) = i\}$ and consider the associated sequence $\mathcal{P}(0, 1) = (\eta_1, \eta_2, \dots)$.

Lemma 5.2. *The sequence $\mathcal{P}(0, 1)$ is a partition, i.e., $\eta_1 \geq \eta_2 \geq \dots$. In addition, $\eta_1 > \eta_2$.*

Proof. This is a particular case of a more general observation, see [16, Prop. 3.1]. However, that proof consists of a reference to case-by-case and computer computations. For this reason, we provide a general case-free proof in the Appendix, see Proposition A.1.

Note also that, for the standard gradings, the inequality $\eta_1 > \eta_2$ readily stems from the fact that $\Delta(0)^+ \cup \Delta(1)$ contains all simple roots, i.e., $\eta_1 = \text{rk } \Delta$. \square

Conjecture 5.3. *The arrangement $\mathcal{A}_\Delta(0, 1)$ is free and its exponents are given by the dual partition $\mathcal{P}(0, 1)^t$ to $\mathcal{P}(0, 1)$.*

This is a special case of a general conjecture discussed in [16]. Namely, let $\mathcal{I} \subset \Delta^+$ be an arbitrary upper ideal and $\mathcal{A}_\Delta(\mathcal{I}^c) = \{\mathcal{H}_\gamma \mid \gamma \notin \mathcal{I}\} \subset \mathcal{A}_\Delta$. Sommers and Tymoczko conjecture that the arrangement $\mathcal{A}_\Delta(\mathcal{I}^c)$ is free and its exponents are given by the dual partition to $(\lambda_1, \lambda_2, \dots)$, where $\lambda_i = \#\{\gamma \in \Delta^+ \setminus \mathcal{I} \mid \text{ht}(\gamma) = i\}$. (The string $(\lambda_1, \lambda_2, \dots)$ is really a partition, see Proposition A.1.) By [16, Theorem 11.1], this general conjecture, and thereby Conjecture 5.3, are true if Δ is of type $\mathbf{A}_n, \mathbf{B}_n, \mathbf{C}_n, \mathbf{D}_n$, and \mathbf{G}_2 . Using this conjecture, one derives a closed formula for the number of lower ideals (antichains) in $\Delta(1)$.

Theorem 5.4. *It follows from Conjecture 5.3 that*

$$(5.1) \quad \#(\mathcal{J}_-(\Delta(1))) = \#\mathfrak{A}n(\Delta(1)) = \prod_{\gamma \in \Delta(1)} \frac{\text{ht}(\gamma) + 1}{\text{ht}(\gamma)}.$$

Proof. Let b_1, \dots, b_n be the exponents of the free arrangement $\mathcal{A}_\Delta(0, 1)$. By the factorisation result of Terao (see [10, Theorem 4.137], the characteristic polynomial of $\mathcal{A}_\Delta(0, 1)$ is $\chi_{(0,1)}(t) = \prod_{i=1}^n (t - b_i)$. By a theorem of Zaslavsky [20], the total number of regions of $\mathcal{A}_\Delta(0, 1)$ equals $(-1)^n \chi_{(0,1)}(-1) = \prod_{i=1}^n (b_i + 1)$. By definition of the dual partition, if $\eta_i = \#\{\gamma \in \Delta(0)^+ \cup \Delta(1) \mid \text{ht}(\gamma) = i\}$, then $\eta_i - \eta_{i+1}$ is the number of exponents that are equal to i . Therefore,

$$\prod_{i=1}^n (b_i + 1) = \prod_{\gamma \in \Delta(0)^+ \cup \Delta(1)} \frac{\text{ht}(\gamma) + 1}{\text{ht}(\gamma)}.$$

Since the arrangement $\mathcal{A}_\Delta(0, 1)$ is $W(0)$ -invariant and $\mathcal{C}(0)$ is a fundamental domain for the $W(0)$ -action, the number of regions inside $\mathcal{C}(0)$ equals $\prod_{i=1}^n (b_i + 1) / \#W(0)$. On the other hand, the Kostant-Macdonald identity (3.1) implies that $\#W(0) = \prod_{\gamma \in \Delta(0)^+} \frac{\text{ht}(\gamma) + 1}{\text{ht}(\gamma)}$.

Combining all these formulae, we conclude that the number of regions of $\mathcal{A}_\Delta(0, 1)$ inside $\mathbb{C}(0)$ equals $\prod_{\gamma \in \Delta(1)} \frac{\text{ht}(\gamma)+1}{\text{ht}(\gamma)}$. Finally, by Theorem 5.1(i), the last number also gives the number of antichains (ideals) in $\Delta(1)$. \square

Remark 5.5. Formula (5.1) for $\#\mathfrak{An}(\Delta(1))$ appears already in [13] as a consequence of a general conjectural formula for $\mathcal{M}_{\Delta(1)}(t)$ [13, Conj. 5.1]. Now, our theory of minimal/maximal elements in W^0 , a relationship to arrangements, and partial results of [16] allow us to conclude that (5.1) holds for all classical cases and \mathbf{G}_2 . However, the present approach does not provide new information on $\mathcal{M}_{\Delta(1)}(t)$, because there seems to be no relationship between the arrangement $\mathcal{A}_\Delta(0, 1)$ and the rank-generating function $\mathcal{M}_{\Delta(1)}(t)$.

Example 5.6. In the abelian case, we have $\mathcal{A}_\Delta(0, 1) = \mathcal{A}_\Delta$ and the exponents of the Coxeter arrangement \mathcal{A}_Δ are the usual exponents of the Weyl group W [10, Theorem 6.60]. Hence $\chi_{\mathcal{A}_\Delta}(t) = \prod_{i=1}^n (t - m_i)$ and $(-1)^n \chi_{\mathcal{A}_\Delta}(-1) = \prod_{i=1}^n (m_i + 1) = \#W$, as required.

As usual, we arrange the exponents in the non-decreasing order: $1 = m_1 \leq m_2 \leq \dots \leq m_n = h - 1$. If $n \geq 2$, then $m_1 < m_2$ and $m_{n-1} < m_n$.

Example 5.7. In the extra-special case, $W(0)$ is the stabiliser of θ and $\mathcal{A}_\Delta(0, 1)$ is just the *deleted arrangement* $\mathcal{A}' = \mathcal{A}_\Delta \setminus \mathcal{H}_\theta$. It is known that \mathcal{A}' is free and the exponents of \mathcal{A}' are $m_1, \dots, m_{n-1}, m_n - 1$ (combine Theorems 4.51 and 6.104 in [10]). Therefore $(-1)^n \chi_{\mathcal{A}'}(-1) = (m_1 + 1) \dots (m_{n-1} + 1) m_n = \#W \cdot \frac{h-1}{h}$. Since $\#W/\#W(0)$ is the number of long roots in Δ , the number of the $W(0)$ -dominant regions of \mathcal{A}' is

$$\frac{\#W}{\#W(0)} \cdot \frac{h-1}{h} = \#\Pi_l \cdot h \cdot \frac{h-1}{h} = \#\Pi_l \cdot (h-1),$$

which is the number of antichains in $\Delta(1)$. This was computed earlier in [13, Section 4], see also Example 3.12.

Example 5.8. For the 1-standard \mathbb{Z} -grading of $\mathfrak{g} = \mathbf{E}_7$ with $\Pi(1) = \{\alpha_7\}$, we have $\mathfrak{g}(0) \simeq \mathfrak{gl}(7)$ and $\mathfrak{g}(1) = \wedge^3(\mathbb{C}^7)$ is the third fundamental representation. Here the numbering of Π follows [19, Tables]. Then

$$\mathcal{P}(0, 1) = (7, 6^4, 5^2, 4^2, 3, 2, 1, 1) \text{ and } \mathcal{P}(0, 1)^t = (13, 11, 10, 9, 7, 5, 1).$$

Therefore, the conjectural exponents of $\mathcal{A}_\Delta(0, 1)$ are 1, 5, 7, 9, 10, 11, 13 and then the number of lower ideals in $\Delta(1)$ is 252.

6. AFFINE VERSUS FINITE THEORY

In this section, we compare the theory of upper (or ad-nilpotent) ideals of Δ^+ (the *affine theory*) and our theory of lower ideals in $\Delta(1)$ related to a \mathbb{Z} -grading of Δ (the *finite theory*).

We begin with the necessary notation. Recall that $V = \oplus_{i=1}^n \mathbb{R}\alpha_i$ and (\cdot, \cdot) is a W -invariant inner product on V . As usual, $\mu^\vee = 2\mu/(\mu, \mu)$ is the coroot for $\mu \in \Delta$ and $\mathcal{Q}^\vee = \oplus_{i=1}^n \mathbb{Z}\alpha_i^\vee$ is the *coroot lattice* in V . Letting $\widehat{V} = V \oplus \mathbb{R}\delta \oplus \mathbb{R}\lambda$, we extend the inner product (\cdot, \cdot) on \widehat{V} so that $(\delta, V) = (\lambda, V) = (\delta, \delta) = (\lambda, \lambda) = 0$ and $(\delta, \lambda) = 1$. Set $\alpha_0 = \delta - \theta$.

Then

$$\begin{aligned}\widehat{\Delta} &= \{\Delta + k\delta \mid k \in \mathbb{Z}\} \text{ is the set of affine (real) roots;} \\ \widehat{\Delta}^+ &= \Delta^+ \cup \{\Delta + k\delta \mid k \geq 1\} \text{ is the set of positive affine roots;} \\ \widehat{\Pi} &= \Pi \cup \{\alpha_0\} \text{ is the corresponding set of affine simple roots.}\end{aligned}$$

For any $\gamma \in \widehat{\Delta}$, the reflection $s_\gamma \in GL(\widehat{V})$ is defined in the usual way, via the extended inner product, and the affine Weyl group, \widehat{W} , is the subgroup of $GL(\widehat{V})$ generated by the reflections s_α , $\alpha \in \widehat{\Pi}$. As is well known, \widehat{W} is also a semi-direct product of W and \mathcal{Q}^\vee [2, 7]. It follows that \widehat{W} has two natural actions:

- (a) the linear action on \widehat{V} ;
- (b) the affine-linear action on V .

Using the linear action, one defines the inversion set $\widehat{N}(w) = \{\gamma \in \widehat{\Delta}^+ \mid w(\gamma) \in -\widehat{\Delta}^+\}$ and the length $\widehat{\ell}(w) = \#\widehat{N}(w)$ for any $w \in \widehat{W}$.

The affine theory is well-developed, and we present below notable correlations with results of this article. An overview of the “affine” results discussed below can also be found in [12, Section 2].

1) By the very definition, $\widehat{\Delta}$ is \mathbb{Z} -graded, with $\widehat{\Delta}(k) = \Delta + k\delta$, $k \in \mathbb{Z}$. Extending our previous terminology to the affine case, one can say that this \mathbb{Z} -grading is 1-standard. The unique affine simple root in $\widehat{\Delta}(1)$ is α_0 and the parabolic subgroup $\widehat{W}(0)$ is just W . Accordingly, the set of minimal length coset representatives is

$$\widehat{W}^0 = \{w \in \widehat{W} \mid w(\alpha) \in \widehat{\Delta}^+ \text{ for all } \alpha \in \Pi\}$$

(such elements of \widehat{W} are called *dominant* in [12].) Let \mathcal{I} be an *upper* ideal of the poset (Δ^+, \preceq) , i.e., $\mathcal{I} \in \mathcal{J}_+(\Delta^+)$. The affine theory gets off the ground when one replaces \mathcal{I} with $\delta - \mathcal{I} = \{\delta - \gamma \mid \gamma \in \mathcal{I}\} \subset \widehat{\Delta}(1)$ and seeks for a characterisation of $\delta - \mathcal{I}$ in terms of \widehat{W} , or rather, in terms of \widehat{W}^0 . Note that $\delta - \mathcal{I}$ becomes a *lower* ideal in the negative part of $\widehat{\Delta}(1) \simeq \Delta$.

2) Given $\mathcal{I} \in \mathcal{J}_+(\Delta^+)$, the first basic result is that there is a unique element $w_{\mathcal{I}, \min} \in \widehat{W}^0$ of **minimal** length such that $\widehat{N}(w_{\mathcal{I}, \min}) \cap \widehat{\Delta}(1) = \delta - \mathcal{I}$. Namely,

$$(6.1) \quad \widehat{N}(w_{\mathcal{I}, \min}) = \bigcup_{k \geq 1} (k\delta - \mathcal{I}^k) = \bigcup_{k \geq 1} (\delta - \mathcal{I})^k.$$

The key point is to prove that the RHS is a bi-convex subset of $\widehat{\Delta}^+$, see [4, Sect. 2]. Hence our Theorem 3.3 is a “finite” analogue of that result. Then the set of minimal elements of \mathcal{I} (called *generators* of \mathcal{I} in [11, 12]), i.e., maximal elements of $\delta - \mathcal{I}$ can be characterised via $w_{\mathcal{I}, \min}$, see [11, Theorem 2.2]. The corresponding “finite” assertion is our Theorem 4.1.

3) Since \widehat{W} and $\widehat{\Delta}$ are infinite, one cannot always provide an element $w_{\mathcal{I},\max} \in \widehat{W}^0$ of **maximal** length such that $\widehat{N}(w_{\mathcal{I},\max}) \cap \widehat{\Delta}(1) = \delta - \mathcal{I}$. Sommers proves [15] that such a maximal element exists if and only if $\mathcal{I} \subset \Delta^+ \setminus \Pi$. In that case, $w_{\mathcal{I},\max}$ can be used for describing the maximal elements of $\Delta^+ \setminus \mathcal{I}$, i.e., the minimal elements of $\widehat{\Delta}(1) \setminus (\delta - \mathcal{I})$, see [15, Cor. 6.3]. Our Theorems 3.3 and 4.2 provide finite analogues of this for *all* lower ideals in $\Delta(1)$.

4) In the finite case, $\Delta(1)$ is the weight poset of a *weight multiplicity free* representation of $\mathfrak{g}(0)$, and the maximal and minimal elements in W^0 exist for all lower ideals. But the adjoint representation of \mathfrak{g} is *not* weight multiplicity free (unless $\mathfrak{g} = \mathfrak{sl}_2$). Therefore, in the affine case, one considers only the weight multiplicity free part of \mathfrak{g} corresponding to Δ^+ . A related disadvantage is that $\Delta^+ \setminus \mathcal{I}$ shouldn't be called a "lower ideal" and that $w_{\mathcal{I},\max}$ does not always exist, see 3) above.

5) Among the advantages of the affine case are the following:

- $\widehat{W} = W \ltimes \mathcal{Q}^\vee$ is a semi-direct product having two related actions (on V and \widehat{V});
- δ is a \widehat{W} -invariant element of \widehat{V} and all the pieces $\widehat{\Delta}(k)$ are isomorphic;

These properties often help in computations and allow to achieve more complete results. On the other hand, an advantage of the finite theory is that both W and $W(0)$ contain the elements of maximal length, which yields a natural involution on W^0 and provides a relationship between W_{\min}^0 and W_{\max}^0 in Proposition 3.9.

6) There are at least two approaches to computing the total number of upper ideals (antichains) in Δ^+ , which are discussed below.

(6a) There is a natural bijection between $\mathcal{J}_+(\Delta^+)$ and the W -dominant regions of the *Catalan arrangement*

$$\text{Cat}(\Delta) = \{\mathcal{H}_{\gamma,k} \mid \gamma \in \Delta^+, k = -1, 0, 1\},$$

where $\mathcal{H}_{\gamma,k} = \{v \in V \mid (\gamma, v) = k\}$. Then an explicit formula for the characteristic polynomial of $\text{Cat}(\Delta)$ yields a formula for $\#\mathcal{J}_+(\Delta^+)$, see [1]. A finite counterpart of this approach is implemented in Section 5, $\text{Cat}(\Delta)$ being replaced with $\mathcal{A}_\Delta(0, 1)$. In particular, the "finite" analogue of the above bijection is our Theorem 5.1.

(6b) There is a natural bijection between the set of minimal elements in \widehat{W}^0 , denoted \widehat{W}_{\min}^0 , and the points of certain convex polytope $\mathcal{D}_{\min} \subset V$ lying in \mathcal{Q}^\vee [5, Prop. 3]. This polytope is \widehat{W} -conjugate to a dilated fundamental alcove of \widehat{W} , and the number

$$\#(\mathcal{D}_{\min} \cap \mathcal{Q}^\vee) = \#\widehat{W}_{\min}^0 = \#\mathcal{J}_+(\Delta^+)$$

can be computed via a result of Haiman, see [5, Section 3] for details. To construct a bijection $\widehat{W}_{\min}^0 \xrightarrow{1:1} \mathcal{D}_{\min} \cap \mathcal{Q}^\vee$, Cellini and Papi use the semi-direct product structure of \widehat{W} . However, one can notice that the following synthetic procedure works. If $w_{\mathcal{I},\min}$ is defined by (6.1) and $'\ast'$ denotes the affine-linear action of \widehat{W} , then the point of \mathcal{Q}^\vee corresponding to \mathcal{I} is merely $w_{\mathcal{I},\min} \ast 0$.

Warning. Cellini and Papi [4, 5] give the definition of the inversion set $\widehat{N}(w)$ with the inverse of $w \in \widehat{W}$. Therefore, their minimal element corresponding to \mathcal{I} is the inverse of ours, and hence the points of \mathcal{Q}^\vee corresponding to \widehat{W}_{\min}^0 are also different.

Since $W = \widehat{W}(0)$ is the stabiliser of $0 \in V$ w.r.t. the affine-linear action, a finite analogue of the Cellini-Papi bijection is the following. Suppose that a \mathbb{Z} -grading of Δ is 1-standard and $\Pi(1) = \{\tilde{\alpha}\}$. Then $W(0)$ is the stabiliser of the fundamental weight $\varpi_{\tilde{\alpha}}^\vee$ (Section 3) and we need the cardinality of $W_{\min}^0 \cdot \varpi_{\tilde{\alpha}}^\vee$. This subset of the orbit $W \cdot \varpi_{\tilde{\alpha}}^\vee = W^0 \cdot \varpi_{\tilde{\alpha}}^\vee$ is explicitly described, see Theorem 3.13 and Remark afterwards, but we are unable (yet) to infer from this a way to compute the cardinality.

Remark 6.1. There are many other aspects of the affine theory that are not mentioned above. Developing their “finite” counterparts can (and will) be the subject of forthcoming publications.

APPENDIX A. A PARTITION ASSOCIATED WITH AN UPPER IDEAL OF Δ^+

Let \mathcal{I} be an upper ideal of the poset (Δ^+, \preceq) and $\mathcal{I}^c = \Delta^+ \setminus \mathcal{I}$. Define

$$\lambda_i = \#\{\gamma \in \mathcal{I}^c \mid \text{ht}(\gamma) = i\}.$$

Our goal is to give a case-free proof of the following observation, see [16, Prop. 3.1].

Proposition A.1. *The sequence $(\lambda_1, \lambda_2, \dots)$ is a partition of the number of roots of \mathcal{I}^c . That is, $\lambda_1 \geq \lambda_2 \geq \dots$. Moreover, if $\mathcal{I} \neq \Delta^+$, then $\lambda_1 > \lambda_2$.*

Proof. We use some properties of a principal nilpotent element in the corresponding simple Lie algebra \mathfrak{g} . Recall that $\mathfrak{g} = \mathfrak{u} \oplus \mathfrak{t} \oplus \mathfrak{u}^-$ is a fixed triangular decomposition, Δ^+ is the set of \mathfrak{t} -roots in \mathfrak{u} , and \mathfrak{g}_γ is the roots space corresponding to $\gamma \in \Delta$. Take $e = \sum_{\alpha \in \Pi} e_\alpha$, where e_α is a nonzero element of \mathfrak{g}_α . After work of Dynkin and Kostant in 1950’s, it is known that e is a principal nilpotent element of \mathfrak{g} . Specifically, we need the following properties of the centraliser $\mathfrak{z}_{\mathfrak{g}}(e)$ of e :

$$\mathfrak{z}_{\mathfrak{g}}(e) \subset \mathfrak{u} \text{ and } \dim \mathfrak{z}_{\mathfrak{g}}(e) = n = \text{rk } \mathfrak{g}.$$

The \mathfrak{t} -roots in the derived subalgebra $\mathfrak{u}' = [\mathfrak{u}, \mathfrak{u}]$ are exactly the non-simple positive roots, hence \mathfrak{u}' is of codimension n in \mathfrak{u} . Combining the above properties, we see that the mapping $\text{ad}(e) : \mathfrak{u} \rightarrow \mathfrak{u}'$ is onto. Moreover, both vector spaces are graded:

$$\mathfrak{u} = \bigoplus_{i=1}^{h-1} \mathfrak{u}\langle i \rangle \text{ and } \mathfrak{u}' = \bigoplus_{i=2}^{h-1} \mathfrak{u}\langle i \rangle,$$

where $\mathfrak{u}\langle i \rangle = \bigoplus_{\gamma: \text{ht}(\gamma)=i} \mathfrak{g}_\gamma$, and $\text{ad}(e)$ is a homomorphism of degree 1. Let $\mathfrak{c}_{\mathcal{I}} = \bigoplus_{\gamma \in \mathcal{I}} \mathfrak{g}_\gamma$ be the \mathfrak{b} -stable subspace of \mathfrak{u} corresponding to \mathcal{I} . The quotient spaces $\mathfrak{u}/\mathfrak{c}_{\mathcal{I}}$ and $\mathfrak{u}'/(\mathfrak{c}_{\mathcal{I}} \cap \mathfrak{u}')$

inherit the above grading and the commutative diagram

$$\begin{array}{ccc} \mathfrak{u} & \xrightarrow{\text{ad}(e)} & \mathfrak{u}' \\ \downarrow & & \downarrow \\ \mathfrak{u}/\mathfrak{c}_{\mathcal{I}} & \xrightarrow{\text{ad}(e)} & \mathfrak{u}'/(\mathfrak{c}_{\mathcal{I}} \cap \mathfrak{u}') \end{array}$$

shows that the map in the bottom row is also graded surjective, of degree 1. Furthermore, let $\tilde{\mathfrak{u}}\langle i \rangle$ be the component of grade i in $\mathfrak{u}/\mathfrak{c}_{\mathcal{I}}$. Then $\mathfrak{u}/\mathfrak{c}_{\mathcal{I}} = \bigoplus_{i \geq 1} \tilde{\mathfrak{u}}\langle i \rangle$, $\mathfrak{u}'/(\mathfrak{c}_{\mathcal{I}} \cap \mathfrak{u}') = \bigoplus_{i \geq 2} \tilde{\mathfrak{u}}\langle i \rangle$, and $\dim \tilde{\mathfrak{u}}\langle i \rangle = \lambda_i$. Consequently, the graded surjectivity implies that $\lambda_i \geq \lambda_{i+1}$ for all i .

Finally, if $\mathfrak{c}_{\mathcal{I}} \neq \mathfrak{u}$, then $\tilde{\mathfrak{u}}\langle 1 \rangle \neq 0$, and the image of $e \in \mathfrak{u}\langle 1 \rangle \subset \mathfrak{u}$ in $\tilde{\mathfrak{u}}\langle 1 \rangle \subset \mathfrak{u}/\mathfrak{c}_{\mathcal{I}}$ is a *nonzero* element in the kernel of $\text{ad}(e)$. Hence $\lambda_1 > \lambda_2$. \square

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